WHAT SHOULD BE EMPHASIZED AND WHAT OMITTED IN THE HIGH-SCHOOL COURSE IN ALGEBRA?[1]

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The wording of the topic implies that something is wrong at present in the high-school course in algebra. This assumption seems to find justification in the large number of papers on this subject which have been read in recent years before conferences and associations, in the reports which various committees have worked out and presented before representative bodies of teachers, and in the perennial agitation over the failure of the schools to make their pupils measure up to the college-entrance requirements. From the standpoint of the college and university the freshman far too frequently proves a disappointment and a sorrow in view of the time and pains which have been given him by the secondary teacher in his mathematical preparation; and from the standpoint of the usefulness and power which his knowledge of algebra should give the boy, whether he goes to college or not, there seems much to be desired. Opportunity to make any use of algebra within his environment is unknown and unlooked for by the pupil, and when, perchance, such an opportunity suddenly appears, say in the class in physics, he is taken completely by surprise and surrenders at once, especially if s and t are the unknowns instead of x and y.

Assuming then, that serious trouble exists and further, as implied in the wording of the topic, that this trouble is due, partly at least, to overcrowding of subject-matter into the course and to unpedagogical distribution of emphasis upon the various phases of the subject, the question before us is to determine the *proper basis* upon which the omission of topics and the redistri-

[1] Read before the annual High-School Conference at the University of Illinois, November 22, 1907.

bution of emphasis may best be effected. At the very outset we are confronted with a variety of criteria, every one of which has its strong and enthusiastic advocates. In fact it is precisely because of these many strong claims for consideration that the elementary algebra texts have become so overgrown both as to material and as to methods of presentation. It behooves us, therefore, to consider carefully these various claims, and to proceed cautiously in determining our controlling principle in proposing any radical changes.

In the first place, quite independent of what we might think is most desirable or profitable, there are the demands of various examining bodies. While the College-Entrance Examination Board has done much to unify requirements and to put reasonable limitations on the topics demanded, yet at least one important state university in a recent announcement for the algebra requirement in the *first year* of the high school includes the topics: quadratic equations, radicals, theory of exponents, inequalities, ratio, proportion, and progressions. It is implied in this announcement that a complete and rigorous treatment of these topics is not expected in the first year, but one may well question whether the attempt to force a class of fourteen-year-olds through such a *range* of topics in a single year can be justified on any grounds.

Another obstacle in the way of omitting topics thought to be of minor importance at a given stage of progress is the arbitrary local authority which the teacher often meets. Some superintendents and principals are open-minded and ready to try any manifest improvement, but others are quite the reverse. For instance, a teacher in a large school under public control in an important city testifies that no improvement along the line suggested by our topic is possible in that school, because all sections of the algebra class are periodically and arbitrarily examined by the officer in control, the teachers knowing nothing of the questions to be given, but having only a list of prescribed topics to be covered in each period of time between examinations. The difficulties of examination requirements were recently well summed up by a teacher in Chicago who said: I

must prepare my pupils for the normal examinations which emphasize mechanical processes; for examinations for teachers' certificates which emphasize intricate and demonstrational processes; and for the universities which emphasize graphic work and application to mechanics and physics. What shall I do? My desire is to teach my pupils the things which seem *best adapted to their environment* and development, to give them those phases of algebra which are at once practical, useful, and interesting, but my temptation is to secure sets of previous examination questions from these various sources and cram the pupils to pass examinations. My reputation as a teacher and the standing of our school depend upon having our pupils measure up to these tests.

But secondly, suppose the school authorities are enlightened, liberal-minded, and disposed to help on any movement toward real reform—and we are glad to believe that such is the case in a surely and rapidly widening circle—how shall we then secure any unanimity of action? Here is a body of teachers who believe that rapidity, facility, and accuracy in performing the mechanical manipulations of algebra are the most important things. In fact, if we may judge by the widely prevailing practice, is not this the belief of the vast army of teachers? And have they not some good grounds for this belief? Have not many of the textbooks fostered this belief by so arranging the topics that, with the exception of a few simple applications on the opening pages, nothing but *abstract manipulation* of symbols is provided for the pupil for a period of three or four months, during which time he is likely to receive his lasting impressions of the subject? These teachers say with some force: Is not this the phase of algebra which underlies successful work in other branches, both in high school and in college or engineering school, and is not this just what the university wants of our pupils? Do not the examiners praise the candidates of this type whom we send up for admission? And, after all, is not the training in skilful operation and good form the chief thing to be gotten from the study of algebra? Where will the pupils ever use it, especially if they do not go to college, as most of them do

not? Therefore should we not instil in them this mechanical expertness which will serve them well in the store, the shop, the trade, the office, or the profession? Moreover, they say, we find our pupils enjoy this kind of work. Anything that is in the nature of a puzzle appeals to the child and they actually get infatuated with long examples in highest common divisor or in the addition of long strings of literal fractions, especially if they are all fixed so that at the end everything drops to pieces and gives a simple expression as the answer.

We are not at this moment concerned with weighing these arguments. We simply note that they exist and that large numbers of teachers determine for themselves their attitude toward the proposition which is before us today on the basis of this point of view, and to this view they hold with tenacity.

In the third place, there is another large body of teachers who hold a different view. They say that adeptness in the demonstration of principles is the real thing in algebra, and that no clear and intelligent understanding of the subject can come in any other manner. They admit that accurate mechanical work is essential but say that it is entirely subordinate to clear thinking. These teachers would mark a paper very high if it showed sound reasoning, even though it contained mechanical errors, while those in the preceding class would cut it down mercilessly for every numerical mistake. These teachers would say further: Is not the true purpose in the study of mathematics to gain power in thinking, to develop mental muscle, to learn to know when a conclusion follows from a premise, and not to be deceived by fallacious argument? They maintain that these things should be insisted upon quite as much in algebra as in geometry, that otherwise geometry appeals to the student as that branch of mathematics in which we prove things, while algebra is nothing more than mere ciphering. They insist upon the study of the *foundations* of algebra in the first year, that is, they try to compel the student at the start to appreciate a *demonstration* of those things which he already knows more surely than he does after he has tried to prove them, such as, for instance, $a+b=b+a$, $ab=ba$, $a(bc)=(ab)c$, etc. In geometry

these same teachers start a class on the proof that all straight angles are equal and like theorems. In the words of one writer, these demonstrations are insisted upon not so much to give the student confidence in his intuition as to *justify* the confidence which he has. They maintain that since the only time to make foundations for a house is before you build the house, therefore the foundations of algebra and geometry must be studied at the beginning of these subjects. They claim, further, that herein lies the chief cultural value of the study of mathematics. This is the reason for keeping algebra in the curriculum. This is the bulwark which has enabled mathematics to hold its own alongside of the cultural studies. Did not Lincoln and Garfield by the study of mathematics as a demonstrative science charge their minds with power for the great forensic struggles, in which they engaged, and shall we do less for the youth of today? And, finally, they maintain that there is no true enjoyment in the study of mathematics except in discovering the reasons for things and that the pupils of the first year of the high school appreciate this as keenly as the graduate student in the university. They say it is no less than heartless simply to let them cipher and cipher and never taste the true pleasures of arguing from hypothesis to conclusion.

But still a fourth class of teachers will insist that the basis of modification in our present curriculum should be the development of skill in the statement and solution of problems. They say the real business of algebra is to help unravel the mysteries of our environment through the agency of the equation. To be sure, we must know how to manipulate this tool and should, within reasonable bounds, be acquainted with the logic upon which its use rests, but the great thing is to be able to bring things to pass and to find out results by means of it. They argue that it is this phase of algebra that should constitute the true correlation with geometry and with physics. Herein, they say, is the chief complaint against the algebra pupils when they enter these subjects, namely that they cannot use their algebra to solve problems, and the reason, it is affirmed, is that so much time has been spent in the algebra in mere abstract manipulation,

or in abstruse demonstration, that the pupils have received no adequate training in the statement and solutions of problems. These teachers say that real power lies in the ability to interpret phenomena and that this must begin in simple stages with little problems in algebra in which the given data are to be translated into algebraic symbols and the whole formulated as an equation. This power must be developed gradually and persistently throughout the whole algebra course, advancing step by step to the consideration of more involved data and taking up the study of numerous classes of phenomena, especially those having interest and value in themselves or for their practical applications. In this way only, they say, will the algebra pupil gain the power to use his algebra in the study and interpretation of the phenomena of life. And, moreover, they affirm, that in this way only is found true enjoyment in the study of algebra, namely, the joy derived from doing things which have a practical value, and whose bearing upon life and its intepretation can clearly be seen.

From all these varying points of view, for each of which the advocates are numerous and strenuous in their defense, how shall we decide what to eliminate and what to emphasize in the high-school course in algebra? The examination fiend would have us keep *everything* in and spend our restless days and feverish nights concocting new forms of puzzles to spring upon the boys and girls. The expert manipulator would have us gain time by omitting all demonstrations and most of the reading problems, in order that the highest skill in algebraic gymnastics may be obtained. The theorem demonstrator would have us devote our chief attention to the theory and logic of the subject and so would cut out a large amount of mere drill exercises. The problem solver would reduce our books to a mere collection of exercises and problems and upon these he would drill and drill and drill.

Is there any way to reconcile these elements in any one scheme of procedure? There is still one party to this discussion whose case has not yet been considered. The *boy* himself should have a hearing. We have been considering in detail the nature

of the ammunition and the weight of the charge, but what about the composition and caliber of the gun? How much can the average high-school boy hold and assimilate? What phases of algebra are best suited to him at different periods of his progress? What kind of work in algebra will best fit him for life? What ought to be his relation to college-entrance requirements? The consideration of these and similar questions may aid us in throwing light upon the problem before us. This, as I understand it, will be acting in the spirit of the new education, namely, to resolve the study of the curriculum into the study of the boy and girl of today.

From the standpoint of the *pupil* shall we not find all classes of teachers in substantial agreement on the following points:

1. There is a marked difference in the mental and physical development of the average pupil in the first year of the high school and the same pupil in the second year, and a still more marked difference in the third year. If this is so, then the kind and quantity of work in algebra which is appropriate for the first year should differ essentially from that which may properly be given in the second or third years. This principle is by no means always followed and teachers are often deceived by the fact that the first-year pupil will *memorize* demonstrations, rules, and formulae readily and even then fail to comprehend their real significance simply from lack of the necessary maturity of mind, as is plainly shown by the ease with which the comprehension comes a year or two later.

2. There is a marked difference among the topics of algebra with respect to their comprehension and mastery on the part of the average first-year high-school pupil. For example, consider, on the one hand, the operations of algebra as extensions of those of arithmetic and their applications to the solution of linear equations in one or more unknowns, the simpler processes of factoring and their applications to quadratic equations with *rational* roots, the process of square root and its application to the solution of quadratic equations with *real* roots, and to the handling of simple radicals such as computing the altitude of an equilateral triangle in terms of the sides; and, on the other hand,

the theoretical consideration of the commutative, associative, distributive laws, the demonstration of the Euclidean method of finding the highest common divisor, the theory of equivalent equations, the theoretical treatment of fractional and negative exponents, of radicals, of quadratic equations, etc. I doubt if anyone would claim that the topics in the second list and those in the first are in any manner within a like range of comprehension on the part of a first-year high-school pupil.

3. There is a marked difference between the topics of the second list given above and the subject of plane geometry. Those theoretical and complicated parts of the algebra are far more difficult of real comprehension than are the elements of plane geometry. The geometry is concerned with those things which have a tangible and visible existence, so that the demonstrations may be anchored to a concrete foundation; while the advanced portions of the algebra are at best represented in abstract symbolism whose essence is neither tangible nor visible, and which therefore demands a greater degree of maturity on the part of the pupil in order really to grasp it.

If the foregoing propositions relating to the maturity of the boy are accepted, and they seem almost self-evident, then we can readily reach the following corollaries:

a) Not *all the topics* of what is known as high-school algebra should be attempted in the first year—not even though some college or university specifies that they shall be. If only one year can be given to algebra as is the case in some small schools, or as is sometimes the total requirement for graduation from the high school, let that be devoted to those topics and that presentation of topics which are fitted to the pupil at that age, and which have some relation to his environment.

b) If, however, as is usually the case, at least another half-year is given to the subject, let that be, by all means, *after a year of plane geometry,* both for the reasons given above and because the richest applications of the advanced algebra are found in the geometry. In this way the concrete and mensurational geometry of the grades is used freely in the first-year algebra, the elementary algebra is used freely in the geometry and the

geometry is used freely in the advanced algebra; all of which is a long and commendable step toward breaking down the artificial partitions between these subjects. But wherever in the curriculum it occurs let it be understood that this additional half-year of algebra rests on the assumption that the pupil is now able to look at the subject from a different point of view, from that of the first year, with respect to his maturity and reasoning power.

If this contention be admitted then the first-year course cannot properly be planned by simply making a *vertical* cut at some convenient stopping point in the total body of matter known as high-school algebra. There must be a *horizontal* cut. Almost every topic has a portion properly belonging to the first year and another portion belonging with equal certainty to the advanced course. Let us therefore see if we can formulate a guiding principle upon which the choice of work appropriate for the first year can be made, and then by exclusion determine the portion to leave for the advanced course.

Still looking at the question from the standpoint of the pupil, shall we not agree that it is best to present the first-year algebra as far as possible from a concrete point of view and to develop the processes as directly as may be out of those already familiar in arithmetic? If, therefore, it is agreed that the highly complicated and abstract mechanical manipulations, as well as the purely theoretical and demonstrational portions, are to be postponed till the later course, then we have at once what may be used as the guiding principle in selecting topics for omission and emphasis in the first year; namely,

Let the handling of the equation be the central theme and the solution of problems the main business of the course, and around these let the theory and practice of algebra be built.

The application of this principle under proper restrictions will transform the first-year algebra course into a rational, practical, and pedagogical treatment of the subject, one which may be made to meet the essential demands of each of the four classes of advocates described above.

The points to be noted in applying the principle are:

1. The problems must not be merely a miscellaneous collection. They must be selected, grouped, and graded with far more care than is the case in any mere exercise book, for the plan is to build the algebra around the problem, by means of the equation as the instrument. The problems must be interesting, sensible, practical, in order to appeal to the pupils' best impulses. They must lead to answers worth knowing and working for, either in themselves or for application to other problems.

2. A collection of problems alone, however well graded and carefully grouped, is not of itself a sufficient text for the purpose. The principles of algebra are to be made to grow out of problem situation, not as a demonstrational process at this time, but as a practical extension and development of the principles of arithmetic already in hand. Each set of problems is to lead to the formulation of a definite principle, which in turn will underlie another principle to be drawn in like manner from a succeeding group of problems, and so on, until finally a coherent chain of principles binds the whole together. Failure on this point is the failure to strike when the iron is hot; it is to allow the whole process to degenerate into mere problem solving.

3. The problems themselves will not furnish sufficient practice in applying each new principle as it is brought out, but numerous drill exercises will naturally be associated with each group of problems and will have their point and purpose, therefore, self-evident in each case. With these safeguards provided, we shall now see that our guiding principle will lead to results which should satisfy all concerned:

On this plan, the boy learns to appreciate finding a reason for things because, first of all, he sees a *reason for algebra itself,* and in learning here how to use it both in discovering results and in building up a body of principles, he will be prepared in the later course to appreciate the logic and the reasoning *there* used in establishing theorems relating to the facts with which he already has a working acquaintance.

The boy thus learns to appreciate mechanical manipulation because he sees the *need* of accuracy and skill in using the tools of algebra, and when these drill exercises are given in connec-

tion with the problems to which they are applicable he works with zeal and intelligence because he recognizes the importance of the work. For this reason drill exercises become effective in accomplishing their purpose. The boy learns to solve problems in a way that will give him power and insight when he reaches physics, mechanics, engineering, etc. He has not come to the problem at the *end* of a long and artificially constructed chapter of theory, or after weeks or months of blind mechanical manipulation, but he has learned to recognize the problem as the *vital thing,* the only thing which puts life and blood into the whole subject, and he is therefore at home in the problem environment of later science subjects.

By this process the boy may even become an examination expert. At least he will have gone through a régime which gives him possession of his faculties and power to cope with a situation, which are essential in passing examinations. Moreover, in acquiring his drill work in connection with its *uses* he is more likely to be able to retain and to reproduce it on the spur of the moment than he would be if he had simply memorized it like a parrot.

The application of the principle that the equation in its relation to the problem should be the central theme around which the theory and practice of algebra are to be built will readily determine for us a natural order of topics and the proper distribution of emphasis, as well as those topics which should be omitted from a first course and taken in a later course.

1. If the formal material of algebra is to be introduced as needed in handling problems, then literal fractions, long division, and long multiplication of anything more than binomials, and a large part of factoring—topics which usually occupy most of the early months of the first year—will come in the latter part of the year, thus bringing linear equations in one and two unknowns, and fractional equations, with numerical denominators, early in the course, and introducing here a rich body of practical problems involving all sorts of interesting and useful topics, including the straight line graph as a handy tool both in representing data and in solving problems. To one who has not

given it thought, it is a surprise to find how little of the complicated forms of algebra we ever use in solving problems. Why, then, put the beginner through months of gymnastics in these before allowing him to find out what it is all good for? By reversing the process and allowing him to see some of the *interesting* and *useful* side of the subject *early* in the course, he then will be prepared to take up the formal side later in the year and to handle it with intelligence and zeal.

2. This plan of linking the formal work directly with its applications would bring equations solved by factoring immediately after factoring, long division of polynomials in connection with square root, followed by quadratic equations solved by extracting the root, and immediately applied to a body of rich problems in radicals of the second degree, such as arise in mensurational geometry; and it would postpone literal fractions well toward the end of the year.

3. In general, the principle would postpone to the second course all long and complicated work such as H.C.D. and L.C.M. by the division method, involved complex fractions, intricate factoring, the theoretical treatment of exponents, radicals, and quadratic equations. It would emphasize the useful and practical and postpone to the second course the theoretical and demonstrational side. It would carefully develop out of arithmetic, and formulate, the principles of algebra and make them familiar by means of constant application to useful and practical things.

4. In the second course, when the pupil is more mature, can be emphasized the demonstration of principles and the deeper consideration of the topics either not touched upon in the earlier course or only treated in an elementary manner. In the second course all the requisites of college entrance can be fully met.

If the question arises as to the status of the first year's work, as above outlined, with respect to college requirements, we may safely affirm that what is best for the boy in first-year algebra ought to be recognized as good enough for a unit of college admission, and what is best for college admission in the second course in algebra ought to be recognized as good for the boy

also, whether he goes to college or not, so that *all* high-school students should find in the whole year and a half of algebra that which is interesting, cultural, and practically useful.

The principle here advocated is by no means new but it has undergone a long period of disuse in this country. The signs are now hopeful, however, of a speedy and effective revival. In fact, scores and even hundreds of schools throughout the country are awakening to the realization that the algebra course has been abstract, barren, and unrelated to the experiences of life, and are making heroic efforts to infuse life-blood into it. But they find themselves limited by the rigidity of existing curricula and especially by the fact that the prevailing textbooks have emphasized the abstract, mechanical, and formal side of the subject. Not a few schools, both large and small, have, however, broken entirely with the traditions, and numerous important steps are in progress looking toward further developments.

For instance, an important report of the State Teachers' Association of Wisconsin which was published this year, is directly in line with the spirit of the suggestions here made. A committee report of the high-school teachers of mathematics in the city of Chicago recommends a reorganization of the algebra work on this basis.

A representative committee of the Central Association of Mathematics Teachers has been working during the past year on a report of far-reaching importance which was presented at the meeting last Autumn in St. Louis, and which has recently been published in *School Science and Mathematics.*

It is no mere coincidence that these and other representative bodies are so strongly advocating these points of view. They are simply evidences of the present trend, and expressions of deepest thought in the concerted and widespread effort to improve the teaching of mathematics. It starts high up and reaches down deep. It is affecting colleges, universities, and technical schools, on the one hand, and the grammar grades on the other hand. The teaching of college algebra, analytics, and calculus is undergoing the same careful scrutiny and reorganization. No more significant fact has appeared on the horizon

than the publication this year of a book called *A Course in Mathematics,* by Professors Woods and Baily of the Massachusetts Institute of Technology, which, if I understand it, is intended to revitalize the first two years of college mathematics in a manner comparable to that which has been set forth in this paper for the preparatory algebra, and which the writer has it in mind to advocate for geometry. In fact, it should be said that these present proposals which naturally involve the infusion of geometry into algebra and in turn the infusion of algebra into geometry, will logically lead in the not distant future to *A Course in Mathematics* for the secondary school which shall be as coherent and concrete as the one now prepared for the Institute of Technology.